Trajectory equations under a central force with a perturbations force

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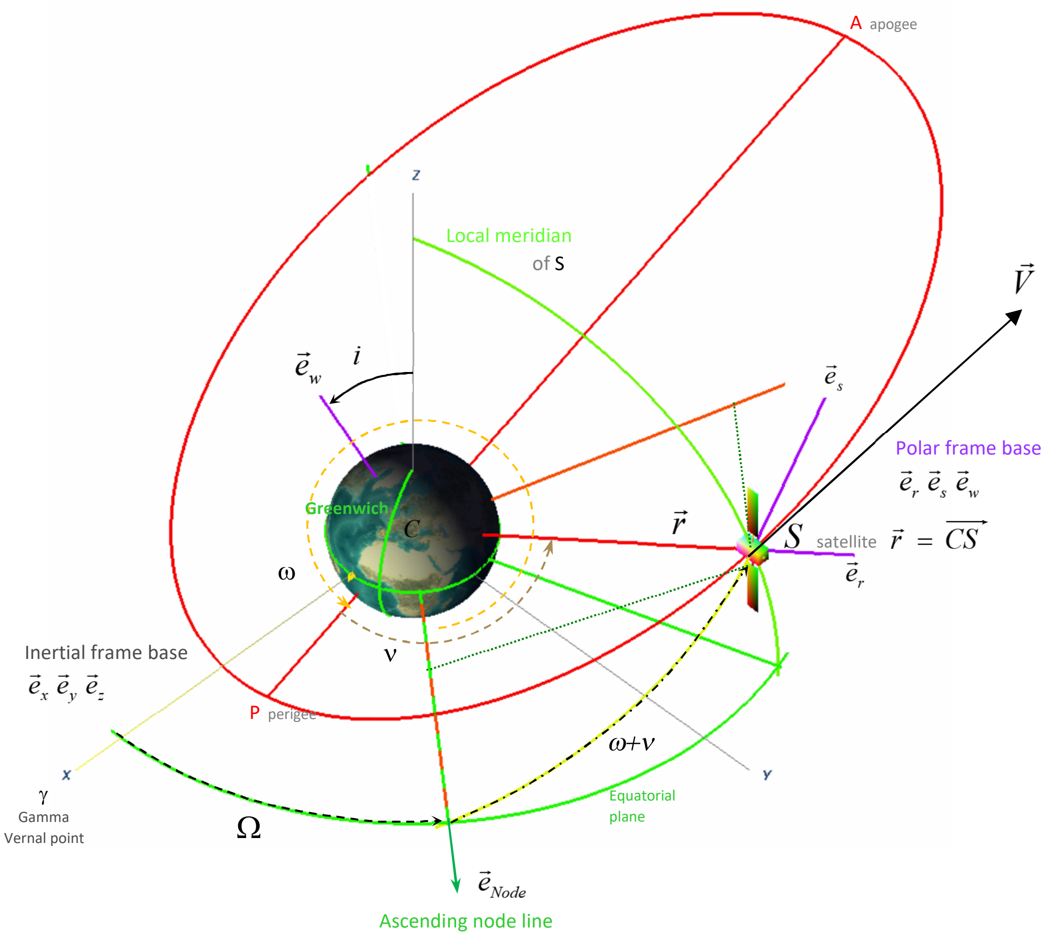
1 Orbital trajectory equations (no perturbations)



The orbital trajectory under a central force $\ddot{\vec{r}} = \frac{-\mu}{r^3}\vec{r}$ is fixed at each time t by the vectors $\vec{r}(t), \vec{V}(t)$.

With $\vec{r}(t), \vec{V}(t)$ one can derive the 5 fixed elliptical parameters: a, e, i, Ω, ω plus the time dependent $\nu(t)$

- It is fully known that the vis viva equation (orbital energy conservation) is $\frac{V^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a}$. It gives the semi-major axis *a (half the distance perigee-apogee).*

- With the orbital angular momentum defined by the cross product $\vec{C} = \vec{r} \times \vec{V}$ along $\vec{e}_w$ and $C^2 = \mu.a(1-e^2)$ one gets the eccentricity *e*, the polar equation of the ellipse $r(1+e.\cos\nu) = a(1-e^2)$ and its radial derivative $\mu.r.e.\sin\nu = C\ \vec{V}\cdot\vec{r}$. Hence $\cos\nu$ and $\sin\nu$ give the full determination of true anomaly ν with the Atan2() function.

  The inclination angle $i \in [0,\pi]$ and right ascension of the ascending node Ω are defined by the vector $\vec{C}$

  $\vec{C}\cdot\vec{e}_z = C\cos i$ With $i \in [0,\pi]$, the determination of i is complete simply with the function Acos()

  $$\left.\begin{aligned}\vec{C}\cdot\vec{e}_x &= C\cos\left(\Omega - \pi/2\right)\sin i\\ \vec{C}\cdot\vec{e}_y &= C\sin\left(\Omega - \pi/2\right)\sin i\end{aligned}\right\}$$

  Give the full determination of Ω with the Atan2() function (for $\sin i \neq 0$)

- The perigee argument ω is given by the polar equation projected on the node line and then on the inertial frame:

  $$\vec{r}\cdot\vec{e}_x = r\cos(\omega+\nu)\cos\Omega + r\sin(\omega+\nu)\cos i\cos\left(\Omega+\pi/2\right)$$
  $$\vec{r}\cdot\vec{e}_y = r\cos(\omega+\nu)\sin\Omega + r\sin(\omega+\nu)\cos i\sin\left(\Omega+\pi/2\right)$$
  $$\vec{r}\cdot\vec{e}_z = +r\sin(\omega+\nu)\sin i$$
  $$\left.\begin{aligned}&\Rightarrow \vec{r}\cdot\vec{e}_x\cos\Omega + \vec{r}\cdot\vec{e}_y\sin\Omega = r\cos(\omega+\nu)\\ &\Rightarrow r\sin(\omega+\nu) = \vec{r}\cdot\vec{e}_z/\sin i\end{aligned}\right\}$$

  Give the full determination of $\omega+\nu$ with the Atan2() function (for $\sin i \neq 0$)

- The mean anomaly is given for t after t_{per} the time crossing the perigee

  $M(t) = \sqrt{\mu/a^3}(t - t_{per}) = \sqrt{\mu/a^3}(u(t) - e\sin u(t))$ where u is the eccentric anomaly (angle from center *O* wrt perigee) given by $\mathrm{tg}\frac{\nu(t)}{2}) = \sqrt{\frac{1+e}{1-e}}\mathrm{tg}\frac{u(t)}{2}$ coming from $r = a(1-e\cos u(t))$.

2 Orbital trajectory equations with perturbation force

With perturbation force $\vec{F}$ force per unit of mass i.e. $\vec{F}$ is acceleration $\ddot{\vec{r}} = \frac{d\vec{V}}{dt} = \frac{-\mu}{r^3}\vec{r} + \vec{F}$

The summary of chapter 5 is shown first.

$$\vec{C} = \vec{r}\times\vec{V} \quad \frac{d\vec{C}}{dt} = \vec{r}\times\vec{F} \quad h = \frac{V^2}{2} - \frac{\mu}{r} \quad \frac{dh}{dt} = \vec{V}\cdot\vec{F} \quad \vec{E} = \frac{1}{\mu}\vec{V}\times\vec{C} - \frac{\vec{r}}{r} \quad \frac{d\vec{E}}{dt} = \frac{1}{\mu}\left[\vec{F}\times\vec{C} + \vec{V}\times(\vec{r}\times\vec{F})\right]$$

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a} \qquad \frac{da}{dt} = \frac{2a^2}{\mu}\vec{V}\cdot\vec{F} \qquad \frac{dC}{dt} = \frac{1}{C}\vec{C}\times\vec{r}\cdot\vec{F}$$

$$e^2 = 1 - \frac{C^2}{\mu a} \qquad \frac{de}{dt} = \frac{C^2}{\mu^2 e}\vec{V}\cdot\vec{F} - \frac{\vec{C}\cdot(\vec{r}\times\vec{F})}{\mu ea} \qquad \frac{de}{dt} = \frac{1}{\mu e}\left[\vec{F}\times\vec{C} + \vec{V}\times(\vec{r}\times\vec{F})\right]\cdot\vec{E}$$

$$\vec{C}\cdot\vec{e}_z = C\cos i \qquad \frac{di}{dt} = \frac{r\cos(\omega+\nu)}{C^2}\vec{C}\cdot\vec{F}$$

$$\mathrm{tg}\Omega = \frac{\vec{C}\cdot\vec{e}_x}{-\vec{C}\cdot\vec{e}_y} \qquad \frac{d\Omega}{dt} = \frac{r\sin(\omega+\nu)}{C^2\sin i}\vec{C}\cdot\vec{F}$$

$$\frac{d\omega}{dt} = \frac{1}{\mu Ce^2}\left[\vec{F}\times\vec{C} + \vec{V}\times(\vec{r}\times\vec{F})\right]\cdot\vec{C}\times\vec{E} - \cos i\frac{d\Omega}{dt}$$

$$\frac{dM}{dt} \approx M(t)/(t - t_{per}) + \frac{-1}{\sqrt{\mu a}}\left[2\vec{r}\cdot\vec{F} + C\frac{d\omega}{dt} + C\cos i\frac{d\Omega}{dt}\right]$$

3 Application: best command to increase the perigee, [R1]

$$r_{per} = a(1-e) \qquad \frac{dr_{per}}{dt} = (1-e)\dot{a} - a\dot{e}$$

$$\frac{dr_{per}}{dt} = \frac{\vec{C}\cdot(\vec{r}\times\vec{F})}{\mu e} - \frac{(1-e)^2 a^2}{\mu e}\vec{V}\cdot\vec{F}$$

To increase the perigee at best, the thrust shall be in the local horizontal plane (i.e. along $\vec{C}\times\vec{r}$ in order to get $\vec{C}\cdot(\vec{r}\times\vec{F})$ maximum) and around apogee (i.e. $\vec{r}$ maximum and $\vec{V}\cdot\vec{F}$ minimum).

Note: this suggestion made to ESOC has been used in the frame of SMART-1 for many orbits to escape as fast as possible the Van Allen belts (maximum perigee increase).

4 Application: best command to increase the apogee, [R1]

$$r_{apo} = a(1+e) \qquad \frac{dr_{apo}}{dt} = (1+e)\dot{a} + a\dot{e}$$

$$\frac{dr_{apo}}{dt} = \frac{(1+e)^2 a^2}{\mu e}\vec{V}\cdot\vec{F} - \frac{\vec{C}\cdot(\vec{r}\times\vec{F})}{\mu e}$$

To increase the apogee at best, the thrust shall be along velocity (i.e. $\vec{V}\cdot\vec{F}$ maximum) and around perigee (i.e. $\vec{V}$ maximum and $\vec{C}\cdot(\vec{r}\times\vec{F})$ minimum).

5 Development of Gauss equations [R2] , [R3] , [R4]

Note: all derivative wrt an inertial frame, thus $\vec{e}_x, \vec{e}_y, \vec{e}_z$ are fixed vector $\Rightarrow \dot{\vec{e}}_x, \dot{\vec{e}}_y, \dot{\vec{e}}_z = 0$

$$\ddot{\vec{r}} = \frac{d\vec{V}}{dt} = \frac{-\mu}{r^3}\vec{r} + \vec{F} \qquad \text{nota : } \vec{V} = \frac{d\overrightarrow{cs}}{dt} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = \frac{d(r\vec{e}_r)}{dt} = \frac{dr}{dt}\vec{e}_r + \mathrm{r}\frac{d\vec{e}_r}{dt} = \dot{r}\vec{e}_r + \mathrm{r}\dot{\vec{e}}_r = \dot{r}\vec{e}_r + \mathrm{r}\dot{\nu}\,\vec{e}_s$$

with ν the true anomaly

$$\vec{C} = \vec{r}\times\vec{V} \qquad \frac{d\vec{C}}{dt} = \frac{d\vec{r}}{dt}\times\vec{V} + \vec{r}\times\frac{d\vec{V}}{dt} = \vec{V}\times\vec{V} + \vec{r}\times\frac{-\mu}{r^3}\vec{r} + \vec{r}\times\vec{F} = 0 + 0 + \vec{r}\times\vec{F} \qquad \frac{d\vec{C}}{dt} = \vec{r}\times\vec{F}$$

$$h = \frac{V^2}{2} - \frac{\mu}{r} \qquad \text{for elliptical orbits} \quad h = \frac{-\mu}{2a}$$

$$\frac{V^2}{2} = \frac{1}{2}\frac{d\dot{r}^2}{dt} = \ddot{\vec{r}}\cdot\dot{\vec{r}} = \left(\frac{-\mu}{r^3}\vec{r} + \vec{F}\right)\cdot\dot{\vec{r}} = \frac{-\mu}{r^3}\vec{r}\cdot\dot{\vec{r}} + \vec{F}\cdot\dot{\vec{r}} = \frac{-\mu}{r^3}\vec{r}\cdot\dot{\vec{r}} + \vec{V}\cdot\vec{F}$$

$$\text{and} \quad -\frac{d\frac{\mu}{r}}{dt} = \frac{d\mu(\vec{r}\cdot\vec{r})^{-1/2}}{dt} = \mu\frac{1}{2}(\vec{r}\cdot\vec{r})^{-3/2}.2(\vec{r}\cdot\dot{\vec{r}}) = \frac{\mu}{r^3}\vec{r}\cdot\dot{\vec{r}} \qquad \frac{dh}{dt} = \vec{V}\cdot\vec{F}$$

$$\text{Hence,} \quad \frac{dh}{dt} = \frac{d\left(\frac{\dot{r}^2}{2} - \frac{\mu}{r}\right)}{dt} = \frac{1}{2}\frac{d\dot{r}^2}{dt} - \frac{d\frac{\mu}{r}}{dt} = \vec{V}\cdot\vec{F}$$

$$a = \frac{-\mu}{2h}$$

$$\frac{da}{dt} = \frac{\mu}{2h^2}\cdot\frac{dh}{dt} = \frac{\mu}{2h^2}\vec{V}\cdot\vec{F} = \frac{2a^2}{\mu}\vec{V}\cdot\vec{F} \qquad \frac{da}{dt} = \frac{2a^2}{\mu}\vec{V}\cdot\vec{F}$$

$$\vec{E}=\frac{1}{\mu}\vec{V}\times\vec{C}-\frac{\vec{r}}{r} \qquad \frac{d\vec{E}}{dt}=\frac{1}{\mu}\frac{d\vec{V}}{dt}\times\vec{C}+\frac{1}{\mu}\vec{V}\times\frac{d\vec{C}}{dt}-\dot{\vec{e}}_r=\frac{1}{\mu}\left[\frac{-\mu}{r^3}\vec{r}+\vec{F}\right]\times\vec{C}+\frac{1}{\mu}\vec{V}\times(\vec{r}\times\vec{F})-\dot{\vec{e}}_r$$

$$\frac{d\vec{E}}{dt}=\frac{1}{\mu}\left[\vec{F}\times\vec{C}+\vec{V}\times(\vec{r}\times\vec{F})\right]-\left(\frac{\vec{r}}{r^3}\times(\vec{r}\times\vec{V})+\dot{\vec{e}}_r\right)=\frac{1}{\mu}\left[\vec{F}\times\vec{C}+\vec{V}\times(\vec{r}\times\vec{F})\right]+0$$

because $\frac{\vec{r}}{r^3}\times(\vec{r}\times\vec{V})+\dot{\vec{e}}_r=\frac{\vec{r}\cdot\vec{V}}{r^3}\vec{r}-\frac{\vec{r}\cdot\vec{r}}{r^3}\vec{V}+\dot{\vec{e}}_r=\frac{\vec{e}_r\cdot\vec{V}}{r}\vec{e}_r-\frac{\vec{V}}{r}+\dot{\vec{e}}_r=\frac{\dot{r}}{r}\vec{e}_r+\underbrace{\cancel{(\vec{e}_r\cdot\dot{\vec{e}}_r)}}_{=0}\vec{e}_r-\frac{\dot{r}}{r}\vec{e}_r-\dot{\vec{e}}_r+\dot{\vec{e}}_r=0$

$$\boxed{\frac{d\vec{E}}{dt}=\frac{1}{\mu}\left[\vec{F}\times\vec{C}+\vec{V}\times(\vec{r}\times\vec{F})\right]}$$

Useful memo

$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{c}\cdot\vec{a})\vec{b}-(\vec{b}\cdot\vec{a})\vec{c}$ (the ABC is caB baC)

but $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{c}\cdot\vec{b})\vec{a}$

$(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})$ (from Jacobi identity)

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=(\vec{a}.\vec{c})(\vec{b}\cdot\vec{d})-(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})$ (Binet-Cauchy identity)

$\left[=\vec{d}\cdot(\vec{a}\times\vec{b})\times\vec{c}=\vec{d}\cdot\left((\vec{a}\cdot\vec{c})\vec{b}-(\vec{c}\cdot\vec{b})\vec{a}\right)\right]$ (It transforms cross products into scalar products so, null in case of orthogonal)

$\vec{C}=C\vec{e}_w$ with Euler angles rotation $\vec{\Omega}_{R/R_0}=\frac{d\Omega}{dt}\vec{e}_z+\frac{di}{dt}\vec{e}_{Node}+\frac{dw}{dt}\vec{e}_w$

$$\frac{d\vec{C}}{dt}=\frac{dC}{dt}\vec{e}_w+\vec{\Omega}_{R/R_0}\times\vec{C}=\frac{dC}{dt}\vec{e}_w+C\left(\frac{d\Omega}{dt}\vec{e}_z+\frac{di}{dt}\vec{e}_{Node}+\frac{dw}{dt}\vec{e}_w\right)\times\vec{e}_w=\frac{dC}{dt}\vec{e}_w+C\frac{d\Omega}{dt}\vec{e}_z\times\vec{e}_w+C\frac{di}{dt}\vec{e}_{Node}\times\vec{e}_w$$

$\frac{d\vec{C}}{dt}\cdot\vec{e}_w=\frac{dC}{dt}$ $\qquad \frac{d\vec{C}}{dt}=\vec{r}\times\vec{F}$ $\qquad \frac{dC}{dt}=\vec{r}\times\vec{F}\cdot\vec{e}_w=r\vec{e}_r\times\vec{F}\cdot\vec{e}_w$ $\qquad \boxed{\frac{dC}{dt}=r\vec{e}_s\cdot\vec{F}}$ $\qquad \vec{e}_s=\frac{1}{rC}\vec{C}\times\vec{r}$

$\frac{d\vec{C}}{dt}\cdot\vec{e}_{Node}=C\frac{d\Omega}{dt}\vec{e}_z\times\vec{e}_w\cdot\vec{e}_{Node}=C\frac{d\Omega}{dt}\sin i$ $\qquad C\frac{d\Omega}{dt}\sin i=r\vec{e}_r\times\vec{F}\cdot\vec{e}_{Node}$ $\qquad \boxed{\frac{d\Omega}{dt}=\frac{r\sin(\omega+\nu)}{C\sin i}\vec{e}_w\cdot\vec{F}}$

because $\vec{e}_{Node}\times\vec{e}_r=\sin(\omega+\nu)\vec{e}_w$ $\qquad \vec{e}_w=\frac{1}{C}\vec{C}$

$\frac{d\vec{C}}{dt}\cdot\vec{e}_{Node}\times\vec{e}_w=C\frac{d\Omega}{dt}\vec{e}_z\times\underbrace{\cancel{\vec{e}_w\cdot\vec{e}_{Node}}}_{=0}\times\vec{e}_w+C\frac{di}{dt}$ $\qquad \frac{di}{dt}=\frac{r}{C}\vec{e}_r\times\vec{F}\cdot\vec{e}_{Node}\times\vec{e}_w$ $\qquad \frac{di}{dt}=\frac{r}{C}(\vec{e}_w\times\vec{e}_{Node})\times\vec{e}_r\cdot\vec{F}$

$$\boxed{\frac{di}{dt}=\frac{r\cos(\omega+\nu)}{C}\vec{e}_w\cdot\vec{F}}$$

$\vec{E}=e\vec{e}_{per}$ with $\vec{e}_{per}$ the unit vector focus to perigee

$$\frac{d\vec{E}}{dt}=\frac{de}{dt}\vec{e}_{per}+\vec{\Omega}_{R/R_0}\times\vec{E}=\frac{de}{dt}\vec{e}_{per}+e\frac{d\Omega}{dt}\vec{e}_z\times\vec{e}_{per}+e\frac{di}{dt}\vec{e}_{Node}\times\vec{e}_{per}+e\frac{dw}{dt}\vec{e}_w\times\vec{e}_{per}$$

$\frac{d\vec{E}}{dt}\cdot\vec{e}_{per}=\frac{de}{dt}=\frac{1}{\mu}\left[\vec{F}\times\vec{C}+\vec{V}\times(\vec{r}\times\vec{F})\right]\cdot\vec{e}_{per}$ $\qquad \boxed{\frac{de}{dt}=\frac{1}{\mu e}\left[\vec{F}\times\vec{C}+\vec{V}\times(\vec{r}\times\vec{F})\right]\cdot\vec{E}}$

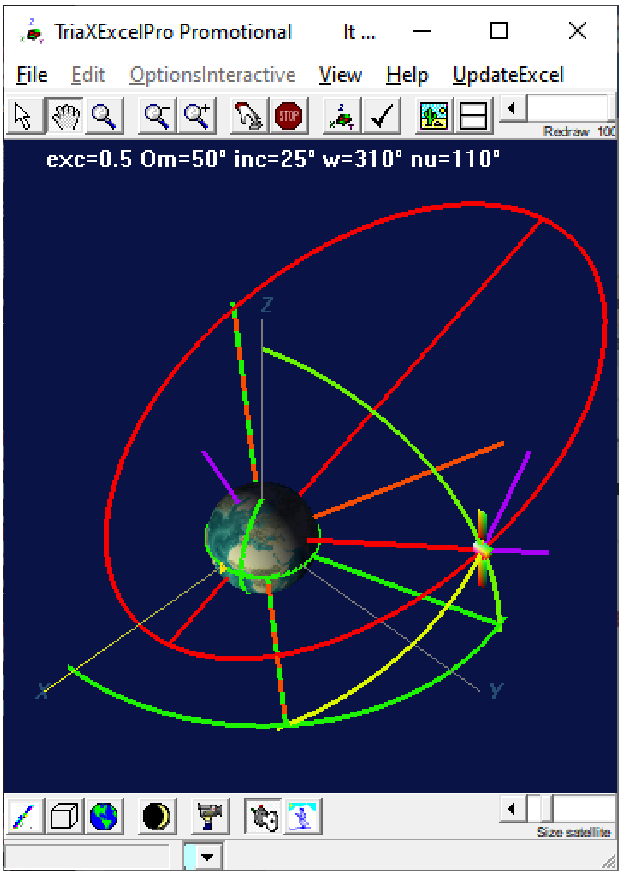
$$\frac{d\vec{E}}{dt}\cdot\vec{e}_w\times\vec{e}_{per}=e\frac{d\Omega}{dt}\vec{e}_z\times\vec{e}_{per}\cdot\vec{e}_w\times\vec{e}_{per}+e\frac{dw}{dt}=e\frac{d\Omega}{dt}\cos i+e\frac{dw}{dt}=\frac{1}{\mu}\left[\vec{F}\times\vec{C}+\vec{V}\times(\vec{r}\times\vec{F})\right]\cdot\vec{e}_w\times\vec{e}_{per}$$

because $\vec{e}_z\times\vec{e}_{per}\cdot\vec{e}_w\times\vec{e}_{per}=(\vec{e}_w\times\vec{e}_{per})\times\vec{e}_z\cdot\vec{e}_{per}=-\vec{e}_z\times(\vec{e}_w\times\vec{e}_{per})\cdot\vec{e}_{per}=-\vec{e}_z.\vec{e}_{per}\vec{e}_w\cdot\vec{e}_{per}+\vec{e}_z.\vec{e}_w\vec{e}_{per}\cdot\vec{e}_{per}=\vec{e}_z.\vec{e}_w=\cos i$

$$\boxed{\cos i\frac{d\Omega}{dt}+\frac{dw}{dt}=\frac{1}{\mu Ce^2}\left[\vec{F}\times\vec{C}+\vec{V}\times(\vec{r}\times\vec{F})\right]\cdot\vec{C}\times\vec{E}}$$

6 References:

[R 1] Christophe Koppel, Thrust Orientation Strategies for a Continuous Transfer to GEO, LOTUS conference, ATELIER CNES "TRAJECTOIRES A POUSSEE FAIBLE" 7-8 mars 2000, Toulouse
[R 2] Luc Duriez, Cours de Mécanique céleste classique, Laboratoire d'Astronomie de l'Université de Lille 1 et IMCCE de l'Observatoire de Paris, 2002
[R 3] Pierre Exertier, Florent Deleflie, Les Equations du Mouvement Orbital Perturbé, Cours de l'Ecole GRGS 2002, Observatoire de la Côte d'Azur (CERGA/URA6527), Av. N. Copernic, F-06130 Grasse, 21 février 2003
[R 4] KopooS, TriaXOrbitaL, TriaXExcelPro tools 1989-2021
[R 5] P. Duchon, J.M. Guilbert, L. Marechal, "Stabilisation des satellites," 1983 Supaero.



La mécanique orbitale est une discipline étrange... La première fois que vous la découvrez, vous ne comprenez rien... La deuxième fois, vous pensez que vous comprenez, sauf un ou deux points... La troisième fois, vous savez que vous ne comprenez plus rien, mais à ce niveau vous êtes tellement habitué que ça ne vous dérange plus. attribué à Arnold Sommerfeld pour la thermodynamique, vers 1940

7 Annex: Other approaches for Gauss equations development: [R 3]

$$\frac{de}{dt}=\frac{1}{\mu e}\left[\vec F\times\vec C+\vec V\times(\vec r\times\vec F)\right]\cdot\left[\frac{1}{\mu}\vec V\times\vec C-\frac{\vec r}{r}\right]=\frac{1}{\mu^2 e}\left(\vec F\times\vec C\cdot\vec V\times\vec C+\vec V\times(\vec r\times\vec F)\cdot\vec V\times\vec C\right)-\frac{1}{\mu e r}\left(\vec F\times\vec C\cdot\vec r+\vec V\times(\vec r\times\vec F)\cdot\vec r\right)$$

$$\vec F\times\vec C\cdot\vec r+\vec V\times(\vec r\times\vec F)\cdot\vec r=\vec r\times\vec F\cdot\vec C+\vec r\times\vec V\cdot\vec r\times\vec F=2\vec C\cdot\vec r\times\vec F$$

$$\frac{de}{dt}=\frac{C^2}{\mu^2 e}\vec F\cdot\vec V+\frac{V^2}{\mu^2 e}\vec C\cdot\vec r\times\vec F-\frac{2}{\mu e r}\vec C\cdot\vec r\times\vec F=\frac{C^2}{\mu^2 e}\vec F\cdot\vec V+\frac{2}{\mu^2 e}\left(\frac{V^2}{2}-\frac{\mu}{r}\right)\vec C\cdot\vec r\times\vec F=\frac{C^2}{\mu^2 e}\vec F\cdot\vec V+\frac{2}{\mu^2 e}\left(\frac{-\mu}{2a}\right)\vec C\cdot\vec r\times\vec F$$

$$\boxed{\frac{de}{dt}=\frac{C^2}{\mu^2 e}\vec V\cdot\vec F-\frac{\vec C\cdot\left(\vec r\times\vec F\right)}{\mu e a}}$$

$$e^2=1-\frac{C^2}{\mu a}\qquad e^2=\vec E\cdot\vec E=1-\frac{\vec C\cdot\vec C}{\mu a}\qquad \frac{de}{dt}=\frac{d\left(\vec E\cdot\vec E\right)^{1/2}}{dt}=\frac{1}{2}\left(\vec E\cdot\vec E\right)^{-1/2}2\vec E\cdot\frac{d\vec E}{dt}=\frac{1}{e}\vec E\cdot\frac{d\vec E}{dt}$$

$$2e\frac{de}{dt}=\frac{d\left(-\frac{\vec C\cdot\vec C}{\mu a}\right)}{dt}=\frac{-2\vec C\cdot\frac{d\vec C}{dt}}{\mu a}+\frac{\vec C\cdot\vec C}{\mu a^2}\frac{da}{dt}=\frac{-2\vec C\cdot\left(\vec r\times\vec F\right)}{\mu a}+\frac{C^2}{\mu a^2}\frac{2a^2}{\mu}\vec V\cdot\vec F=\frac{-2\vec C\cdot\left(\vec r\times\vec F\right)}{\mu a}+\frac{2C^2}{\mu^2}\vec V\cdot\vec F$$

$$\frac{de}{dt}=\frac{C^2}{\mu^2 e}\vec V\cdot\vec F-\frac{\vec C\cdot\left(\vec r\times\vec F\right)}{\mu e a}$$

$\vec{C}\cdot\vec{e}_z = C\cos i$ in the inertial frame, $\vec{e}_z$ is the unit vector along z (fixed vector $\dot{\vec{e}}_z = 0$)

$$\vec{e}_z = \sin(w+v)\sin i\,\vec{e}_r + \cos(w+v)\sin i\,\vec{e}_s + \cos i\,\vec{e}_w$$

$$\frac{d\vec{C}\cdot\vec{e}_z}{dt} = \frac{d\vec{C}}{dt}\cdot\vec{e}_z = \frac{d\left(\vec{C}\cdot\vec{C}\right)^{\frac{1}{2}}}{dt}\cos i - C\sin i\frac{di}{dt} = \frac{1}{2}\left(\vec{C}\cdot\vec{C}\right)^{\frac{-1}{2}}2\vec{C}\cdot\frac{d\vec{C}}{dt}\cos i - C\sin i\frac{di}{dt}$$

$$\frac{di}{dt} = \frac{\frac{\vec{C}}{C}\cdot\frac{d\vec{C}}{dt}\cos i - \frac{d\vec{C}}{dt}\cdot\vec{e}_z}{C\sin i} = \frac{\frac{\cos i}{C}\vec{C} - \vec{e}_z}{C\sin i}\cdot\frac{d\vec{C}}{dt} = \frac{\cos i\vec{e}_w - \vec{e}_z}{C\sin i}\cdot\vec{r}\times\vec{F}$$

$$\frac{di}{dt} = \frac{q_{\text{any}}\vec{e}_r + \cos(\omega+v)\sin i\,\vec{e}_s}{-C\sin i}\cdot\vec{r}\times\vec{F},$$ i.e. because the scalar triple product $\vec{e}_r\cdot\vec{r}\times\vec{F} = 0$

$$\frac{di}{dt} = \frac{-\cos(\omega+v)}{C}\vec{e}_s\cdot\vec{r}\times\vec{F} = \frac{r\cos(\omega+v)}{C}\vec{e}_w\cdot\vec{F}$$

$$\frac{di}{dt} = \frac{r\cos(\omega+v)}{C}\vec{e}_w\cdot\vec{F}$$

$$\vec{C} = C\sin\Omega\sin i\vec{e}_x - C\cos\Omega\sin i\vec{e}_y + C\cos i\vec{e}_z \qquad \mathrm{tg}\Omega = \frac{\vec{C}\cdot\vec{e}_x}{-\vec{C}\cdot\vec{e}_y} \Rightarrow \frac{d\mathrm{tg}\Omega}{dt} = \frac{1}{\cos^2\Omega}\frac{d\Omega}{dt}$$ and

$$\mathrm{d}\frac{\vec{C}\cdot\vec{e}_x}{-\vec{C}\cdot\vec{e}_y}\Big/dt = -\frac{1}{C^2\cos^2\Omega\sin^2 i}\frac{d\vec{C}}{dt}\cdot(\vec{C}\cdot\vec{e}_y\vec{e}_x - \vec{C}\cdot\vec{e}_x\vec{e}_y) = \frac{1}{C\cos^2\Omega\sin i}\frac{d\vec{C}}{dt}\cdot(\cos\Omega\vec{e}_x + \sin\Omega\vec{e}_y)$$

$$\frac{d\Omega}{dt} = \cos^2\Omega\frac{d\mathrm{tg}\Omega}{dt} = \frac{1}{C\sin i}\frac{d\vec{C}}{dt}\cdot(\cos\Omega\vec{e}_x + \sin\Omega\vec{e}_y) = \frac{1}{C\sin i}\vec{r}\wedge\vec{F}\cdot(\cos\Omega\vec{e}_x + \sin\Omega\vec{e}_y)$$

with $\vec{e}_x = q_x\vec{e}_r + \left(-\cos(\omega+\nu)\sin\Omega\cos i - \sin(\omega+\nu)\cos\Omega\right)\vec{e}_s + \sin\Omega\sin i\vec{e}_w$

and $\vec{e}_y = q_y\vec{e}_r + \left(+\cos(\omega+\nu)\cos\Omega\cos i - \sin(\omega+\nu)\sin\Omega\right)\vec{e}_s - \cos\Omega\sin i\vec{e}_w$

$$q_x = -\sin(\omega+v)\sin\Omega\cos i + \cos(\omega+v)\cos\Omega$$
$$q_y = \sin(\omega+v)\cos\Omega\cos i + \cos(\omega+v)\sin\Omega$$

$\cos\Omega\vec{e}_x + \sin\Omega\vec{e}_y = q_{any}\vec{e}_r - \sin(\omega+\nu)\vec{e}_s$; the $\vec{e}_r$ component is cancelled in the scalar triple product

$$\frac{d\Omega}{dt} = \frac{-\sin(\omega+\nu)}{C\sin i}\vec{e}_s\cdot\vec{r}\times\vec{F} = \frac{r\sin(\omega+\nu)}{C\sin i}\vec{e}_w\cdot\vec{F}$$

$$\frac{d\Omega}{dt} = \frac{r\sin(\omega+\nu)}{C\sin i}\vec{e}_w\cdot\vec{F}$$

8 Annex: Case of circular or/and equatorial orbits

TBC
